On densest packings of equal balls of $\mathbb{R}^n$ and Marcinkiewicz spaces

Gilbert Muraz and Jean-Louis Verger-Gaugry

Institut Fourier - CNRS UMR 5582, Université Grenoble I,
BP 74 - Domaine Universitaire,
38402 - Saint Martin d'Hères, France
email: `jlverger@ujf-grenoble.fr`

Abstract. We investigate, by " à la Marcinkiewicz" techniques applied to the (asymptotic) density function, how dense systems of equal spheres of $\mathbb{R}^n, n \geq 1$, can be partitioned at infinity in order to allow the computation of their density as a true limit and not a limsup. The density of a packing of equal balls is the norm 1 of the characteristic function of the systems of balls in the sense of Marcinkiewicz. Existence Theorems for densest sphere packings and completely saturated sphere packings of maximal density are given new direct proofs.

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Contents

1 Introduction

The existence of densest sphere packings in $\mathbb{R}^n, n \geq 2$, asked the question to know how they could be constructed. The problem of constructing very dense sphere packings between the bounds of Kabatjanskii-Levenstein and Minkowski-Hlawka type bounds (see Fig. 1 in [MVG1]) remains open [Bz] [Ca] [CS] [GL] [GOR] [R] [Z]. There are two problems: the first one is the determination of the supremum δ_n over all possible densities, δ_n being called the packing constant, as a function of n only (for $n = 3$ see Hales [H]); the second one consists in characterizing the (local, global) configuration of balls in a densest sphere packing, namely for which the density is δ_n.

The notion of complete saturation was introduced by Fejes-Toth, Kuperberg and Kuperberg [FTKK]. Section 2 gives new direct proofs of the existence Theorems for completely saturated sphere packings (see Bowen [Bo] for a proof with $\mathbb{R}^n$ and $\mathbb{H}^n$ as ambient spaces) of maximal density and densest sphere packings in $\mathbb{R}^n$. For this purpose new metrics are introduced (Subsection 2.1) on the space of uniformly discrete sets (space of equal sphere packings), and this leads to a continuity Theorem for the density function (Theorem 7.2).

Let Λ be a uniformly discrete set of $\mathbb{R}^n$ of constant $r > 0$, that is a discrete point set for which $\|x - y\| \geq r$ for all $x, y \in \Lambda$, with equality at least for one couple of elements of Λ, and consider the system of spheres (in fact *balls*) $\mathcal{B}(\Lambda) = \{\lambda + B(0, \frac{r}{2}) \mid \lambda \in \Lambda\}$, where $B(c, t)$ denotes the closed ball of center c and radius t. Let $B = B(0, 1/2)$. The fact that the density

$$\delta(\mathcal{B}(\Lambda)) := \limsup_{T \to +\infty} \left[\mathrm{vol}\big(\big(\bigcup_{\lambda \in \Lambda} (\lambda + B(0, r/2)) \big) \bigcap B(0, T) \big) / \mathrm{vol}(B(0, T)) \right]$$

of $\mathcal{B}(\Lambda)$ is equal to the norm ("norm 1") of Marcinkiewicz of the characteristic function $\chi_{\mathcal{B}(\Lambda)}$ of $\mathcal{B}(\Lambda)$ [B+] [PH] [M], namely

$$\delta(\mathcal{B}(\Lambda)) = \|\chi_{\mathcal{B}(\Lambda)}\|_1, \tag{1.1}$$

where, for all $p \in \mathbb{R}^{+*}$ and all $f \in \mathcal{L}^p_{loc}$ with $\mathcal{L}^p_{loc}$ the space of complex-valued functions f defined on $\mathbb{R}^n$ whose p-th power of the absolute value $|f|^p$ is integrable over any bounded measurable subset of $\mathbb{R}^n$ for the Lebesgue measure,

$$\|f\|_p := \limsup_{t\to+\infty} |f|_{p,t}, \tag{1.2}$$

with

$$|f|_{p,t} := \left(\frac{1}{\mathrm{vol}(tB)}\int_{tB}|f(x)|^p dx\right)^{1/p}, \qquad f \in \mathcal{L}^p_{loc}, \tag{1.3}$$

asks the following question: what can tell the theory of Marcinkiewicz spaces to the problem of constructing very dense sphere packings ? Obviously the problem of the determination of the packing constant or more generally of the density is associated with the quotient space $\mathcal{L}^p_{loc}/\mathcal{R}$ where $\mathcal{R}$ is the Marcinkiewicz equivalence relation (Section 3): the density function is a class function, that is is well defined on the Marcinkiewicz space $\mathcal{M}^p$ with $p = 1$. For instance any finite cluster of spheres has the same density, equal to zero, as the empty packing (no sphere); the Marcinkiewicz class of the empty sphere packing being much larger than the set of finite clusters of spheres. Then it suffices to understand the construction of one peculiar sphere packing per Marcinkiewicz class. It is the object of this note to precise the geometrical constraints given by such a construction.

Since any non-singular affine transformation T on a system of balls $\mathcal{B}(\Lambda)$ leaves its density invariant (Theorem 1.7 in [R]), namely

$$\delta(\mathcal{B}(\Lambda)) = \delta(T(\mathcal{B}(\Lambda))), \tag{1.4}$$

we will only consider packings of spheres of common radius 1/2 in the sequel. It amounts to consider the space $\mathcal{UD}$ of uniformly discrete subsets of $\mathbb{R}^n$ of constant 1. Its elements will be called $\mathcal{UD}$-sets. Denote by $\overline{f}$ the class in $\mathcal{M}^p = \mathcal{L}^p_{loc}/\mathcal{R}$ of $f \in \mathcal{L}^p_{loc}$, where $\mathcal{L}^p_{loc}$ is endowed with the $\mathcal{M}^p$-topology (Section 3), and by

$$\nu: \begin{array}{ccc} \mathcal{UD} & \to & \mathcal{L}^1_{loc}, \\ \Lambda & \to & \chi_{\mathcal{B}(\Lambda)} \end{array} \qquad \text{resp.} \quad \overline{\nu}: \begin{array}{ccc} \mathcal{UD} & \to & \mathcal{M}^1 \\ \Lambda & \to & \overline{\chi_{\mathcal{B}(\Lambda)}} \end{array}$$

the (set-) embedding of $\mathcal{UD}$ in $\mathcal{L}^1_{loc}$, resp. in $\mathcal{M}^1$.

Theorem 1.1. *The image $\nu(\mathcal{UD})$ in $\mathcal{L}^1_{loc}\cap\mathcal{L}^\infty$, resp. $\overline{\nu}(\mathcal{UD})$ in $\mathcal{M}^1$, is closed.*

Theorem 1.1 is a reformulation of the following more accurate theorem, since $\mathcal{M}^p$ is complete [B] [B+]. For $0 \leq \lambda \leq \mu$ denote

$$\mathcal{C}(\lambda, \mu) := \{x \in \mathbb{R}^n \mid \lambda \leq \|x\| \leq \mu\}$$

the closed annular region of space between the spheres centered at the origin of respective radii λ and μ.

Theorem 1.2. *Let $(\Lambda_m)_{m\geq 1}$ be a sequence of $\mathcal{UD}$-sets such that the sequence $(\chi_{\mathcal{B}(\Lambda_m)})_{m\geq 1}$ is a Cauchy sequence for the pseudo-metric $\|\cdot\|_1$ on $\mathcal{L}^1_{loc} \cap \mathcal{L}^\infty$. Then, there exist*

(i) *a strictly increasing sequence of positive integers $(m_i)_{i\geq 1}$,*

(ii) *a strictly increasing sequence of real numbers $(\lambda_i)_{i\geq 1}$ with $\lambda_i \geq 1$ and $\lambda_{i+1} > 2\lambda_i$,*

such that, with

$$\Lambda = \bigcup_{i\geq 1} \Lambda_{m_i} \cap \mathcal{C}(\lambda_i + 1/2, \lambda_{i+1} - 1/2), \tag{1.5}$$

the two functions

$$\chi_{\mathcal{B}(\Lambda)} \text{ and } \lim_{i\to+\infty} \chi_{\mathcal{B}(\Lambda_{m_i})}$$

are $\mathcal{M}^1$-equivalent. As a consequence

$$\delta(\mathcal{B}(\Lambda)) \; = \; \lim_{i\to+\infty} \delta(\mathcal{B}(\Lambda_{m_i})). \tag{1.6}$$

The situation is the following for a (densest) sphere packing $\mathcal{B}(\Lambda)$ of $\mathbb{R}^n$ for which $\delta(\mathcal{B}(\Lambda)) = \delta_n$:

* either it cannot be reached by a sequence of sphere packings such as in Theorem 1.2, in which case there is an *isolation phenomenon*,
* or there exists at least one sequence of sphere packings such as in Theorem 1.2, and it is Marcinkiewicz - equivalent to a sphere packing having the asymptotic annular structure given by Theorem 1.2, where the sequence of thicknesses of the annular portions exhibit an exponential growth.

The sharing of space in annular portions as given by Theorem 1.2 may allow constructions of very dense packings of spheres layer-by-layer in each portion independently, since the intermediate regions $\mathcal{C}(\lambda_i - 1/2, \lambda_i + 1/2)$ are all of constant thickness 1 which is twice the common ball radius 1/2. These intermediate regions do not contribute to the density so that they can be filled up or not by spheres. However the existence of such unfilled spherical gaps are not likely to provide completely saturated packings, at least for $n = 2$ [KKK].

Note that the value 2 which controls the exponential sequence of radii $(\lambda_i)_i$ by $\lambda_{i+1} > 2\lambda_i$ in Theorem 1.2 (ii) can be replaced by any value $a > 1$. This is

important for understanding constructions of sphere packings iteratively on the dimension n: indeed, chosing $a > 1$ sufficiently small brings the problem back to fill up first one layer in a as dense as possible way, therefore in dimension $n-1$, then propagating towards the orthogonal direction exponentially.

The terminology *density* is usual in the field of lattice sphere packings, while the terminology *asymptotic measure*, therefore *asymptotic density*, is usual in Harmonic Analysis, both meaning the same in the present context.

2 Densest sphere packings and complete saturation

The set SS of systems of equal spheres of radius $1/2$ and the set $\mathcal{UD}$ are in one-to-one correspondence: $\Lambda = (a_i)_{i\in\mathbb{N}} \in \mathcal{UD}$ is the set of sphere centres of $\mathcal{B}(\Lambda) = \{a_i + B \mid i \in \mathbb{N}\} \in SS$. More conveniently we will use the set $\mathcal{UD}$ of point sets of $\mathbb{R}^n$ instead of SS. The subset of $\mathcal{UD}$ of finite uniformly discrete sets of constant 1 of $\mathbb{R}^n$ is denoted by $\mathcal{UD}_f$.

2.1 A metric on $\mathcal{UD}$ invariant by the rigid motions of $\mathbb{R}^n$

Denote by $O(n,\mathbb{R})$ the n-dimensional orthogonal group of $n \times n$ matrices M, i.e. such that $M^{-1} = {}^tM$. A *rigid motion* (or an *Euclidean displacement*) is an ordered pair (ρ, t) with $\rho \in O(n,\mathbb{R})$ and $t \in \mathbb{R}^n$ [Cp]. The composition of two rigid motions is given by $(\rho, t)(\rho', t') = (\rho\rho', \rho(t') + t)$ and the group of rigid motions is the split extension of $O(n,\mathbb{R})$ by $\mathbb{R}^n$ (as a semi-direct product). It is endowed with the usual topology. Theorem 2.1, obtained as a generalization of the Selection Theorem of Mahler [Cy] [GL] [Ma] [Mt], gives the existence of a metric d on $\mathcal{UD}$ [MVG2] which extends the Hausdorff metric on the subspace $\mathcal{UD}_f$. The metric d is not invariant by translation. From it, adding to the construction of d some additional constraints so that it gains in invariant properties (Proposition 2.2 iii) proved in Section 5), a new metric D, invariant by translation and by the group of rigid motions of $\mathbb{R}^n$ (Theorem 2.3 proved in Section 6), can be constructed, giving a new topology to $\mathcal{UD}$, suitable for studying the continuity of the density function (Theorem 7.2).

Theorem 2.1. *The set $\mathcal{UD}$ can be endowed with a metric d such that the topological space $(\mathcal{UD}, d)$ is compact and such that the Hausdorff metric Δ on $\mathcal{UD}_f$ is compatible with the restriction of the topology of $(\mathcal{UD}, d)$ to $\mathcal{UD}_f$.*

Proof. Theorem 1.2 in [MVG2]. □

Proposition 2.2. *There exists a metric d on $\mathcal{UD}$ such that:*

i) *the space* $(\mathcal{UD}, d)$ *is compact,*

ii) *the Hausdorff metric on* $\mathcal{UD}_f$ *is compatible with the restriction of the topology of* $(\mathcal{UD}, d)$ *to* $\mathcal{UD}_f$*,*

iii) $d(\Lambda, \Lambda') = d(\rho(\Lambda), \rho(\Lambda'))$ *for all* $\rho \in O(n, \mathbb{R})$ *and* $\Lambda, \Lambda' \in \mathcal{UD}$.

Since the density of a sphere packing is left invariant by any non-singular affine transformation ((1.4); Theorem 1.7 in Rogers [R]), it is natural to construct metrics on $\mathcal{UD}$ which are at least invariant by the translations and by the orthogonal group of $\mathbb{R}^n$. Such a metric is given by the following theorem.

Theorem 2.3. *There exists a metric* D *on* $\mathcal{UD}$ *such that:*

i) $D(\Lambda_1, \Lambda_2) = D(\rho(\Lambda_1) + t, \rho(\Lambda_2) + t)$ *for all* $t \in \mathbb{R}^n, \rho \in O(n, \mathbb{R}^n)$ *and all* $\Lambda_1, \Lambda_2 \in \mathcal{UD}$,

ii) *the space* $(\mathcal{UD}, D)$ *is complete and locally compact,*

iii) *(pointwise pairing property) for all non-empty* $\Lambda, \Lambda' \in \mathcal{UD}$ *such that* $D(\Lambda, \Lambda') < \epsilon$*, each point* $\lambda \in \Lambda$ *is associated with a unique point* $\lambda' \in \Lambda'$ *such that* $\|\lambda - \lambda'\| < \epsilon/2$*,*

iv) *the action of the group of rigid motions* $O(n, \mathbb{R}) \ltimes \mathbb{R}^n$ *on* $(\mathcal{UD}, D)$: $((\rho, t), \Lambda) \to (\rho, t) \cdot \Lambda = \rho(\Lambda) + t$ *is such that its subgroup of translations* $\mathbb{R}^n$ *acts continuously on* $\mathcal{UD}$.

2.2 Existence Theorems

The two following Theorems rely upon the continuity of the density function $\| \cdot \|_1 \circ \nu$ on the space $(\mathcal{UD}, D)$ (Theorem 2.3 and Theorem 7.2).

Theorem 2.4. *There exists an element* $\Lambda \in \mathcal{UD}$ *such that the following equality holds:*

$$\delta(\mathcal{B}(\Lambda)) \;=\; \delta_n. \tag{2.1}$$

Proof. See Groemer [Gr] and Section 7. □

We will say that $\Lambda \in \mathcal{UD}$ is *saturated*, or *maximal*, if it is impossible to add a replica of the ball B (a ball of radius $1/2$) to $\mathcal{B}(\Lambda)$ without destroying the fact that it is a packing of balls, i.e. without creating an overlap of balls. The set SS of systems of balls of radius $1/2$, is partially ordered by the relation $\prec$ defined by $\Lambda_1, \Lambda_2 \in \mathcal{UD}$, $\mathcal{B}(\Lambda_1) \prec \mathcal{B}(\Lambda_2) \iff \Lambda_1 \subset \Lambda_2$. By Zorn's lemma, maximal packings of balls exist. The saturation operation of a

packing of balls consists in adding balls to obtain a maximal packing of balls. It is fairly arbitrary and may be finite or infinite. More generally [FTKK], $\mathcal{B}(\Lambda)$ is said to be *m-saturated* if no finite subsystem of $m-1$ balls of it can be replaced with m replicas of the ball $B(0, r/2)$. The notion of m-saturation was introduced by Fejes-Toth, Kuperberg and Kuperberg [FTKK]. Obviously, 1-saturation means saturation, and m-saturation implies $(m-1)$-saturation. It is not because a packing of balls is saturated, or m-saturated, that its density is equal to δ_n. The packing $\mathcal{B}(\Lambda)$ is *completely saturated* if it is m-saturated for every $m \geq 1$. Complete saturation is a sharper version of maximum density [Ku].

Theorem 2.5. *Every ball in $\mathbb{R}^n$ admits a completely saturated packing with replicas of the ball, whose density is equal to the packing constant δ_n.*

Proof. Theorem 1.1 in [FTKK]. See also Bowen [Bo]. A direct proof is given in Section 7, where we prove that there always exists a completely saturated sphere packing in the Marcinkiewicz class of a densest sphere packing. $\square$

3 Marcinkiewicz spaces and norms

Let $p \in \mathbb{R}^{+*}$. The Marcinkiewicz p-th space $\mathcal{M}^p$ is the quotient space of the subspace $\{f \in \mathcal{L}^p_{loc} \mid \|f\|_p < +\infty\}$ of $\mathcal{L}^p_{loc}$ by the equivalence relation $\mathcal{R}$ which identifies f and g as soon as $\|f-g\|_p = 0$ (Marcinkiewicz [M], Bertrandias [B], Vo Khac [VK]):

$$\mathcal{M}^p := \{\overline{f} \mid f \in \mathcal{L}^p_{loc}, \|f\|_p < +\infty\}. \tag{3.1}$$

This equivalence relation is called Marcinkiewicz equivalence relation. It is usual to introduce, with $|f|_{p,t}$ given by (1.3), the two semi-norms

$$\|f\|_p := \limsup_{t \to +\infty} |f|_{p,t}$$

and

$$|||f|||_p := \sup_{t>0} |f|_{p,t}$$

on $\mathcal{L}^p_{loc}$. The vector space $\mathcal{M}^p$ is then normed with $\|\overline{f}\|_p = \|f\|_p$.

Theorem 3.1. *The space $\mathcal{M}^p$ is complete.*

Proof. Marcinkiewicz [M], [B], [VK]. $\square$

We call $\mathcal{M}^p$-topology the topology induced by this norm on $\mathcal{M}^p$ or on $\mathcal{L}^p_{loc}$ itself. Both spaces will be endowed with this topology.

Following Bertrandias [B] we say that a function $f \in \mathcal{L}^p_{loc}$ is $\mathcal{M}^p$-regular if

$$\lim_{t\to+\infty} \frac{1}{\mathrm{vol(t\ B)}} \int_{tB\setminus(t-l)B} |f(x)|^p dx = 0 \qquad \text{for all real number } l.$$

Since all functions $f \in \mathcal{L}^p_{loc}$ such that $\|f\|_p = 0$ are $\mathcal{M}^p$-regular, we consider classes of $\mathcal{M}^p$-regular functions of $\mathcal{L}^p_{loc}$ modulo the Marcinkiewicz equivalence relation. We call $\mathcal{M}^p_r$ the set of classes of Marcinkiewicz equivalent $\mathcal{M}^p$-regular functions.

Proposition 3.2. *The set $\mathcal{M}^p_r$ is a complete vector subspace of $\mathcal{M}^p$.*

Proof. [B]. □

4 Proof of Theorem 1.2

Theorem 1.2 is the n-dimensional version of the remark of Marcinkiewicz [M] in the case $p = 1$. We prove a theorem slightly stronger than Theorem 1.2 (Theorem 4.2), by making the assumption in Lemma 4.1 and in Theorem 4.2 that p is ≥ 1 in full generality.

Lemma 4.1. *Let $p \geq 1$. Let $(\lambda_i)_{i\geq 1}$ be a sequence of real numbers such that*

$$\lambda_i \geq 1, \quad \lambda_{i+1} > 2\lambda_i, \qquad\qquad i \geq 1.$$

Let $\mathcal{C}_i := \mathcal{C}\big(\lambda_i + 1/2, \lambda_{i+1} - 1/2\big)$. Then for all bounded function $f \in \mathcal{L}^p_{loc}$ such that $f_{|\mathcal{C}_i} \equiv 0$ for all $i \geq 1$, we have

$$\|f\|_p = 0$$

Proof. Immediate. □

Lemma 4.1 is the special case of $\mathcal{M}^p$-regularity applied to the characteristic functions of systems of spheres which eventually lie within the spherical intermediate regions $\mathcal{C}_i$. It proves that such spheres do not contribute to the density anyway.

Theorem 4.2. *Let $p \geq 1$. Let $(\Lambda_m)_{m\geq 1}$ be a sequence of $\mathcal{UD}$-sets such that the sequence $(\chi_{\mathcal{B}(\Lambda_m)})_{m\geq 1}$ is a Cauchy sequence for the pseudo-metric $\|\cdot\|_p$ on $\mathcal{L}^p_{loc} \cap \mathcal{L}^\infty$. Then, there exist*

(i) *a strictly increasing sequence of positive integers* $(m_i)_{i\geq 1}$,

(ii) *a strictly increasing sequence of real numbers* $(\lambda_i)_{i\geq 1}$ *with* $\lambda_i \geq 1$ *and* $\lambda_{i+1} > 2\lambda_i$,

such that, with

$$\Lambda = \bigcup_{i\geq 1} \Lambda_{m_i} \cap \mathcal{C}(\lambda_i + 1/2, \lambda_{i+1} - 1/2), \tag{4.1}$$

the two functions

$$\chi_{\mathcal{B}(\Lambda)} \qquad \textit{and} \qquad \lim_{i\to+\infty} \chi_{\mathcal{B}(\Lambda_{m_i})} \tag{4.2}$$

are $\mathcal{M}^p$*-equivalent.*

Proof. Since the sequence $(\chi_{\mathcal{B}(\Lambda_m)})$ is a Cauchy sequence, let us chose a subsequence of $\mathcal{UD}$-sets $(\Lambda_{m_i})_{i\geq 1}$ which satisfies

$$\|\chi_{\mathcal{B}(\Lambda_{m_i})} - \chi_{\mathcal{B}(\Lambda_{m_{i+1}})}\|_p \leq 2^{-(i+1)}\,.$$

Then, denoting

$$R_\lambda(f) := \sup_{\lambda+1/2\leq t<+\infty} \left(\frac{1}{\operatorname{vol}(tB)} \int_{tB} |f(x)|^p dx\right)^{1/p}, \qquad f \in \mathcal{L}^p_{loc},$$

let us chose a sequence of real numbers $(\lambda_i)_{i\geq 1}$ for which $\lambda_i \geq 1, \lambda_{i+1} > 2\lambda_i$, and such that

$$R_{\lambda_i}(\chi_{\mathcal{B}(\Lambda_{m_i})} - \chi_{\mathcal{B}(\Lambda_{m_{i+1}})}) \leq 2^{-i}.$$

Let us define the function

$$H(x) := \begin{cases} \chi_{\mathcal{B}(\Lambda_{m_i})}(x) & \text{if} \quad \lambda_i + 1/2 \leq \|x\| \leq \lambda_{i+1} - 1/2 \quad (i = 1, 2, \ldots), \\ 0 & \text{if} \quad \lambda_i - 1/2 < \|x\| < \lambda_i + 1/2 \quad (i = 1, 2, \ldots), \\ 0 & \text{if} \quad \|x\| \leq \lambda_1 - 1/2. \end{cases}$$

The function $H(x)$ is exactly the characteristic function of $\mathcal{B}(\Lambda)$ on $J := \bigcup_{j=1}^{+\infty} \mathcal{C}_j$ the portion of space occupied by the closed annuli $\mathcal{C}_j$. Let us prove that the function $H(x)$ satisfies:

$$\lim_{i\to+\infty} \|H - \chi_{\mathcal{B}(\Lambda_{m_i})}\|_p = 0. \tag{4.3}$$

Let us fix i and take t and k such that

$$\lambda_k + 1/2 \leq 2t \leq \lambda_{k+1} - 1/2 \tag{4.4}$$

holds with $k \geq i + 1$. Then

$$\int_{tB} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx \;=\; \int_{tB\cap J} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx$$

$$+\int_{tB\cap(\mathbb{R}^n\setminus J)} \chi_{\mathcal{B}(\Lambda_{m_i})}(x)^p dx.$$

By Lemma 4.1,

$$\|\chi_{\mathcal{B}(\Lambda_{m_i})} \cap \chi_{\mathbb{R}^n\setminus J}\|_p \quad = \quad 0.$$

Hence, we have just to consider the portion of space occupied by the spheres $\mathcal{B}(\Lambda_{m_i})$ in $tB \cap J$. We have

$$\int_{tB\cap J} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx \;=\; \sum_{\nu=1}^{i} \int_{tB\cap\mathcal{C}_\nu} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx$$

$$+\sum_{\nu=i+1}^{k-1} \int_{tB\cap\mathcal{C}_\nu} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx + \int_{tB\cap\mathcal{C}_k} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx$$

$$= A + E + C.$$

Let us now transform the sum A:

$$\sum_{\nu=1}^{i} \int_{tB\cap\mathcal{C}_\nu} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx = \sum_{\nu=1}^{i} \int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\nu})}(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx.$$

But, for all $\nu \in \{1, 2, \ldots, i-1\}$,

$$\left(\int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\nu})}(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx\right)^{1/p}$$

$$\leq \sum_{\omega=\nu}^{i-1} \left(\int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\omega})}(x) - \chi_{\mathcal{B}(\Lambda_{m_{\omega+1}})}(x)|^p dx\right)^{1/p}$$

$$\leq \sum_{\omega=\nu}^{i-1} (\mathrm{vol}((\lambda_\omega + 1/2)B))^{1/p}\; R_{\lambda_\omega}(\chi_{\mathcal{B}(\Lambda_{m_\omega})} - \chi_{\mathcal{B}(\Lambda_{m_{\omega+1}})})$$

$$\leq \sum_{\omega=\nu}^{i-1} (\mathrm{vol}((\lambda_i + 1/2)B))^{1/p}\; 2^{-\omega} \leq (\mathrm{vol}((\lambda_i + 1/2)B))^{1/p}.$$

Hence

$$A \leq i\,\mathrm{vol}((\lambda_i + 1/2)B). \tag{4.5}$$

Let us transform the sum E:

$$\sum_{\nu=i+1}^{k-1} \int_{tB\cap\mathcal{C}_\nu} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx = \sum_{\nu=i+1}^{k-1} \int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\nu})}(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx.$$

But, for all $\nu \in \{i+1, i+2, \ldots, k-1\}$,

$$\left(\int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\nu})}(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx\right)^{1/p}$$

$$\leq \sum_{\omega=i}^{\nu-1} \left(\int_{tB\cap\mathcal{C}_\nu} |\chi_{\mathcal{B}(\Lambda_{m_\omega})}(x) - \chi_{\mathcal{B}(\Lambda_{m_{\omega+1}})}(x)|^p dx\right)^{1/p}$$

$$\leq \sum_{\omega=i}^{\nu-1} (\mathrm{vol}((\lambda_\omega + 1/2)B))^{1/p}\ R_{\lambda_\omega}(\chi_{\mathcal{B}(\Lambda_{m_\omega})} - \chi_{\mathcal{B}(\Lambda_{m_{\omega+1}})})$$

$$\leq \sum_{\omega=i}^{\nu-1} (\mathrm{vol}((\lambda_\omega + 1/2)B))^{1/p}\ 2^{-\omega} \leq \mathrm{vol}((\lambda_\nu + 1/2)B)^{1/p}\, 2^{-i+1}.$$

Hence,

$$E \leq 2^{-(i-1)p}\,(\mathrm{vol}((\lambda_{i+1} + 1/2)B) + \mathrm{vol}((\lambda_{i+2} + 1/2)B) + \ldots$$

$$+ \mathrm{vol}((\lambda_{k-1} + 1/2)B))$$

$$\leq 2^{-(i-1)p}\,\mathrm{vol}((\lambda_k + 1/2)B)\left(\frac{1}{2^n} + \frac{1}{2^{2n}} + \ldots\right)$$

$$\leq 2^{-(i-1)p}\,\mathrm{vol}(2tB) = 2^{-(i-1)p+n}\,\mathrm{vol}(tB). \tag{4.6}$$

Let us transform the sum C:

$$C \leq 2^{-(i-1)p+n}\,\mathrm{vol}(tB). \tag{4.7}$$

From (4.5), (4.6) and (4.7) we deduce

$$\left(\frac{1}{\mathrm{vol}(tB)} \int_{tB\cap J} |H(x) - \chi_{\mathcal{B}(\Lambda_{m_i})}(x)|^p dx\right)^{1/p}$$

$$\leq \left[\frac{i\,\mathrm{vol}((\lambda_i + 1/2)B)}{\mathrm{vol}(tB)} + 2^{-(i-1)p+n+1}\right]^{1/p}.$$

Using (4.4) we deduce, for a certain constant $c > 0$,

$$\|H - \chi_{\mathcal{B}(\Lambda_{m_i})}\|_p \leq c\,2^{-(i-1)}.$$

This implies (4.3). Now, if $m_i \leq q < m_{i+1}$, $i \geq 1$, we have

$$\|H - \chi_{\mathcal{B}(\Lambda_q)}\|_p \leq \|\chi_{\mathcal{B}(\Lambda_{m_i})} - \chi_{\mathcal{B}(\Lambda_q)}\|_p + \|H - \chi_{\mathcal{B}(\Lambda_{m_i})}\|_p = o(1) + o(1) = o(1)$$

when i tends to $+\infty$. The proof of the $\mathcal{M}^p$-equivalence (4.2) between H and $\lim_{i\to+\infty} \chi_{\mathcal{B}(\Lambda_i)}$ is now complete.

The thickness of the empty annular intermediate regions $\mathcal{C}_i$ is equal to 1: it ensures that the limit point set Λ is uniformly discrete of constant 1. $\square$

5 Proof of Proposition 2.2

The metric d on $\mathcal{UD}$ was constructed in [MVG2], §3.2.1, as a kind of counting system normalized by a suitable distance function. In order to make explicit the statement iii) of Proposition 2.2, we recall the construction of d, adding the ingredient (5.2) in order to obtain the claim. The metric d is given in Lemma 5.1.

For all $\Lambda \in \mathcal{UD}$, we denote by Λ_i its i-th element. Let

$$\mathcal{E} = \{(D, E) \mid D \text{ countable point set in } \mathbb{R}^n, E \text{ countable point set in } (0, 1/2)\}$$

and $f : \mathbb{R}^n \to [0, 1]$ a continuous function with compact support in $B(0, 1)$ which satisfies:

$$f(0) = 1, \tag{5.1}$$

$$f(\rho(t)) = f(t) \qquad \text{for all } t \in \mathbb{R}^n \text{ and all } \rho \in O(n, \mathbb{R}), \tag{5.2}$$

$$f(t) \leq \frac{1/2 + \|\lambda - t/2\|}{1/2 + \|\lambda\|} \qquad \text{for all } t \in B(0, 1) \text{ and } \lambda \in \mathbb{R}^n. \tag{5.3}$$

It is remarkable that the topology of $(\mathcal{UD}, d)$ does not depend upon f once (5.1) and (5.3) are simultaneously satisfied ([MVG2] Proposition 3.5 and §3.3). Therefore adding (5.2) does not change the topology of $(\mathcal{UD}, d)$ but only the invariance properties of the metric d.

For f for instance, let us take $f(t) = 1 - 2\|t\|$ for $t \in B(0, 1/2)$ and $f(t) = 0$ elsewhere.

With each element $(D, E) \in \mathcal{E}$ and origin α of $\mathbb{R}^n$ we associate a real-valued function $d_{\alpha,(D,E)}$ on $\mathcal{UD} \times \mathcal{UD}$ in the following way (denoting by $\overset{\circ}{B}(c, v)$ the interior of the closed ball $B(c, v)$ of centre c and radius $v > 0$). Let $\mathcal{B}_{(D,E)} = \{\mathcal{B}_m\}$ denote the countable set of all possible finite collections

$$\mathcal{B}_m = \{\overset{\circ}{B}(c_1^{(m)}, \epsilon_1^{(m)}), \overset{\circ}{B}(c_2^{(m)}, \epsilon_2^{(m)}), \ldots, \overset{\circ}{B}(c_{i_m}^{(m)}, \epsilon_{i_m}^{(m)})\}$$

of open balls such that $c_q^{(m)} \in D$ and $\epsilon_q^{(m)} \in E$ for all $q \in \{1, 2, \ldots, i_m\}$, and such that for all m and any two distinct balls in $\mathcal{B}_m^{(r)}$ of respective centers

$c_q^{(m)}$ and $c_k^{(m)}$, we have

$$\|c_q^{(m)} - c_k^{(m)}\| \geq 1.$$

Then we define the following function, with $\Lambda, \Lambda' \in \mathcal{UD}$,

$$d_{\alpha,(D,E)}(\Lambda, \Lambda') :=$$

$$\sup_{\mathcal{B}_m \in \mathcal{B}_{(D,E)}} \frac{|\phi_{\mathcal{B}_m}(\Lambda) - \phi_{\mathcal{B}_m}(\Lambda')|}{(1/2 + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \|\alpha - c_2^{(m)}\| + \cdots + \|\alpha - c_{i_m}^{(m)}\|)} \tag{5.4}$$

where the function $\phi_{\mathcal{B}_m}$ is given by

$$\phi_{\mathcal{B}_m}(\Lambda) := \sum_{\overset{\circ}{B}(c,\epsilon) \in \mathcal{B}_m} \sum_i \epsilon f\left(\frac{\Lambda_i - c}{\epsilon}\right),$$

putting $\phi_{\mathcal{B}_m}(\emptyset) = 0$ for all $\mathcal{B}_m \in \mathcal{B}_{(D,E)}$ and all $(D,E) \in \mathcal{E}$ by convention.

Lemma 5.1. *For all $(\alpha, (D,E))$ in $\mathbb{R}^n \times \mathcal{E}$, $d_{\alpha,(D,E)}$ is a pseudo-metric on $\mathcal{UD}$. The supremum $d := \sup_{\substack{\alpha \in \mathbb{R}^n \\ (D,E) \in \mathcal{E}}} d_{\alpha,(D,E)}$ is a metric on $\mathcal{UD}$, valued in $[0,1]$.*

Proof. See Muraz and Verger-Gaugry [MVG2]. □

Let us show that d is invariant by the action of the orthogonal group $O(n, \mathbb{R})$.

Lemma 5.2. *For all $(D,E) \in \mathcal{E}, \alpha \in \mathbb{R}^n, \rho \in O(n,\mathbb{R})$ and $\Lambda, \Lambda' \in \mathcal{UD}$, the following equality holds:*

$$d_{\alpha,(D,E)}(\Lambda, \Lambda') = d_{\rho(\alpha),(\rho(D),E)}(\rho(\Lambda), \rho(\Lambda')).$$

Proof. Let $(D,E) \in \mathcal{E}$ and $\mathcal{B}_m \in \mathcal{B}_{(D,E)}$ with

$$\mathcal{B}_m = \{\overset{\circ}{B}(c_1^{(m)}, \epsilon_1^{(m)}), \overset{\circ}{B}(c_2^{(m)}, \epsilon_2^{(m)}), \ldots, \overset{\circ}{B}(c_{i_m}^{(m)}, \epsilon_{i_m}^{(m)})\}.$$

The following inequalities hold:

$$\|c_q^{(m)} - c_k^{(m)}\| \geq 1 \quad \text{ for all } 1 \leq q, k \leq i_m \text{ with} q \neq k.$$

Let $\rho \in O(n, \mathbb{R})$. The collection $\mathcal{B}_m$ is in one-to-one correspondence with the collection of open balls

$$\mathcal{B}_m^{(\rho)} := \{\overset{\circ}{B}(\rho(c_1^{(m)}), \epsilon_1^{(m)}), \overset{\circ}{B}(\rho(c_2^{(m)}), \epsilon_2^{(m)}), \ldots, \overset{\circ}{B}(\rho(c_{i_m}^{(m)}), \epsilon_{i_m}^{(m)})\} \in \mathcal{B}_{(\rho(D),E)},$$

where the following inequalities

$$\|\rho(c_q^{(m)}) - \rho(c_k^{(m)})\| \geq 1$$

are still true for all $1 \leq q, k \leq i_m$ with $q \neq k$. By (5.2) the following equalities hold:

$$\phi_{\mathcal{B}_m}(\Lambda) = \phi_{\mathcal{B}_m^{(\rho)}}(\rho(\Lambda)).$$

Hence, for a given $\alpha \in \mathbb{R}^n$, by taking the supremum over all the collections $\mathcal{B}_m \in \mathcal{B}_{(D,E)}$ of the following identity:

$$\frac{|\phi_{\mathcal{B}_m}(\Lambda) - \phi_{\mathcal{B}_m}(\Lambda')|}{\frac{1}{2} + \|\alpha\| + \|\alpha - c_1^{(m)}\| + \ldots + \|\alpha - c_{i_m}^{(m)}\|} = \frac{\left|\phi_{\mathcal{B}_m^{(\rho)}}(\rho(\Lambda)) - \phi_{\mathcal{B}_m^{(\rho)}}(\rho(\Lambda')\right|}{\frac{1}{2} + \|\rho(\alpha)\| + \|\rho(\alpha) - \rho(c_1^{(m)})\| + \ldots + \|\rho(\alpha) - \rho(c_{i_m}^{(m)})\|}$$

we deduce the claim. □

By taking now the supremum of $d_{\alpha,(D,E)}(\Lambda, \Lambda')$ over all $\alpha \in \mathbb{R}^n$ and $(D, E) \in \mathcal{E}$ we deduce from Lemma 5.2 that

$$d(\Lambda, \Lambda') = d(\rho(\Lambda), \rho(\Lambda'))$$

for all $\Lambda, \Lambda' \in \mathcal{UD}$ and $\rho \in O(n, \mathbb{R})$ as claimed.

6 Proof of Theorem 2.3

The metric d on $\mathcal{UD}$ (Theorem 2.1) has the advantage to make compact the metric space $(\mathcal{UD}, d)$ but, by the way it is constructed, the disadvantage to use a base point (the origin) in the ambient space $\mathbb{R}^n$. We now remove this disadvantage but the counterpart is that the precompactness of the metric space $\mathcal{UD}$ will be lost. In order to do this, let us first define the new collection of metrics (d_x) on $\mathcal{UD}$ indexed by $x \in \mathbb{R}^n$ by

$$d_x(\Lambda, \Lambda') = d(\Lambda - x, \Lambda' - x), \qquad \Lambda, \Lambda' \in \mathcal{UD}.$$

Let us remark that the metric spaces $(\mathcal{UD}, d_x), x \in \mathbb{R}^n$, are all compact (by Theorem 2.1).

Definition 6.1. Let D be the metric on $\mathcal{UD}$, valued in $[0, 1]$, defined by

$$D(\Lambda, \Lambda') := \sup_{x \in \mathbb{R}^n} d_x(\Lambda, \Lambda'), \qquad \text{for } \Lambda, \Lambda' \in \mathcal{UD}.$$

The metric D is called *the metric of the proximity of points*, or *pp-metric.*

Proof of i): By construction, D is invariant by the translations of $\mathbb{R}^n$. Let us prove its invariance by the orthogonal group $O(n, \mathbb{R})$. Let $\Lambda, \Lambda' \in \mathcal{UD}$ and

$x \in \mathbb{R}^n, \rho \in O(n, \mathbb{R})$. Since

$$d(\Lambda, \Lambda') = d(\rho(\Lambda), \rho(\Lambda'))$$

by Lemma 5.2, we deduce

$$d_x(\Lambda, \Lambda') = d(\Lambda - x, \Lambda' - x) = d(\rho(\Lambda) - \rho(x), \rho(\Lambda') - \rho(x)) = d_{\rho(x)}(\rho(\Lambda), \rho(\Lambda')).$$

Hence,

$$\sup_{x \in \mathbb{R}^n} d_x(\Lambda, \Lambda') = \sup_{x \in \mathbb{R}^n} d_{\rho(x)}(\rho(\Lambda), \rho(\Lambda')).$$

This implies

$$D(\Lambda, \Lambda') = D(\rho(\Lambda), \rho(\Lambda')).$$

Proof of ii): any Cauchy sequence for the pp-metric D is in particular a Cauchy sequence for the metric d_x for all $x \in \mathbb{Q}^n$. But $\mathbb{Q}^n$ is countable. Therefore, from any Cauchy sequence for D, a subsequence which converges for all the metrics $d_x, x \in \mathbb{Q}^n$, can be extracted by a diagonalisation process over all $x \in \mathbb{Q}^n$. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, that

$$\sup_{x \in \mathbb{Q}^n} d_x(\Lambda, \Lambda') = \sup_{x \in \mathbb{R}^n} d_x(\Lambda, \Lambda') \qquad \text{for all } \Lambda, \Lambda' \in \mathcal{UD}_r$$

this subsequence, extracted by diagonalization, also converges for the metric D. This prove the completeness of the metric space $(\mathcal{UD}, D)$.

Proof of iii): we will use the pointwise pairing property of the metrics d_x recalled in the following Lemma.

Lemma 6.2. *Let $x \in \mathbb{R}^n$. Let $\Lambda, \Lambda' \in \mathcal{UD}$ assumed non-empty and define $l_x := \inf_{\lambda \in \Lambda} \|\lambda - x\| < +\infty$. Let $\epsilon \in (0, \frac{1}{1+2l_x})$ and let us assume that $d_x(\Lambda, \Lambda') < \epsilon$. Then, for all $\lambda \in \Lambda$ such that $\|\lambda - x\| < \frac{1-\epsilon}{2\epsilon}$,*

(i) *there exists a unique $\lambda' \in \Lambda'$ such that $\|\lambda' - \lambda\| < 1/2$,*

(ii) *this pairing satisfies the inequality $\|\lambda' - \lambda\| \leq (1/2 + \|\lambda - x\|)\epsilon$.*

Proof. See Proposition 3.6 in [MVG2]. □

Let $0 < \epsilon < 1$ and suppose that $\Lambda, \Lambda' \in \mathcal{UD}$ are non-empty and satisfy $D(\Lambda, \Lambda') < \epsilon$. This implies

$$d_\lambda(\Lambda, \Lambda') < \epsilon \qquad \text{for all } \lambda \in \Lambda.$$

From Lemma 6.2, restricting x to all the elements λ of Λ, we deduce

$$\forall \lambda \in \Lambda, \exists \lambda' \in \Lambda' \text{ (unique) such that } \|\lambda - \lambda'\| < \epsilon/2.$$

This proves the existence of unique pointwise pairings of points and the pointwise pairing property for D.

Proof of iv): let us show that

$$\mathcal{UD} \times \mathbb{R}^n \quad \to \quad \mathcal{UD}$$

$$(\Lambda, t) \quad \to \quad \Lambda + t$$

is continuous. Let $\Lambda_0 \in \mathcal{UD}$ and $t_0 \in \mathbb{R}^n$. First, by the pointwise pairing property given by iii), we deduce

$$\lim_{t \to 0} D(\Lambda_0 + t, \Lambda_0) = 0.$$

Let $0 < \epsilon < 1$. Then, there exists $\eta > 0$ such that

$$|t - t_0| < \eta \implies D(\Lambda_0 + (t - t_0), \Lambda_0) < \epsilon/2.$$

Hence, for all $\Lambda \in \mathcal{UD}$ such that $D(\Lambda, \Lambda_0) < \epsilon/2$ and $t \in \mathbb{R}^n$ such that $|t - t_0| < \eta$, we have:

$$D(\Lambda + t, \Lambda_0 + t_0) = D(\Lambda + (t - t_0), \Lambda_0)$$

$$\leq D(\Lambda + (t - t_0), \Lambda_0 + (t - t_0)) + D(\Lambda_0 + (t - t_0), \Lambda_0)$$

$$= D(\Lambda, \Lambda_0) + D(\Lambda_0 + (t - t_0), \Lambda_0) \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

We deduce the claim.

Proof of (ii) (continuation): let us prove that $(\mathcal{UD}, D)$ is locally compact. The Hausdorff metric Δ is defined on the set $\mathcal{F}(\mathbb{R}^n)$ of the non-empty closed subsets of $\mathbb{R}^n$ as follows:

$$\Delta(\Lambda, \Lambda') := \max \{ \inf\{\epsilon \mid \Lambda' \subset \Lambda + B(0, \epsilon)\}, \inf\{\epsilon \mid \Lambda \subset \Lambda' + B(0, \epsilon)\} \}$$

in particular for $\Lambda, \Lambda' \in \mathcal{UD} \setminus \{\emptyset\}$. $\mathcal{UD} \setminus \{\emptyset\}$ is closed in the complete space $(\mathcal{F}(\mathbb{R}^n), \Delta)$. Then $\mathcal{UD} \setminus \{\emptyset\}$ is complete for Δ. On the space $\mathcal{UD} \setminus \{\emptyset\}$, the two metrics D and Δ are equivalent. The element $\emptyset$ (system of spheres with no sphere) is isolated in $\mathcal{UD}$ for D. Hence, it possesses a neighbourhood (reduced to itself) whose closure is compact. Now, if $\Lambda \in \mathcal{UD} \setminus \{\emptyset\}$ and $0 < \epsilon < 1$, the open neighbourhood $\{\Lambda' \in \mathcal{UD} \mid \Lambda' \subset \Lambda + \overset{\circ}{B}(0, \epsilon)\}$ of Λ admits $\{\Lambda' \in \mathcal{UD} \mid \Lambda' \subset \Lambda + B(0, \epsilon)\}$ as closure which is obviously precompact, hence compact, for D or Δ. We deduce the claim.

7 Proofs of Theorem 2.4 and Theorem 2.5

Assume that there does not exist $\Lambda \in \mathcal{UD}$ such that (2.1) holds. Then, by definition, there exists a sequence $(\Lambda_i)_{i\geq 1}$ such that $\Lambda_i \in \mathcal{UD}$ and

$$\lim_{i\to+\infty} \delta(\mathcal{B}(\Lambda_i)) = \delta_n$$

(as a sequence of real numbers).

Lemma 7.1. *There exists a subsequence $(\Lambda_{i_j})_{j\geq 1}$ of the sequence $(\Lambda_i)_{i\geq 1}$ which converges for D.*

Proof. Indeed, the sequence $(\Lambda_i)_{i\geq 1}$ may be viewed as a sequence in the compact space $(\mathcal{UD}, d_x)$ for any $x \in \mathbb{Q}^n$. Therefore, for all $x \in \mathbb{Q}^n$, we can extract a subsequence from it which converges for the metric d_x. Iterating this extraction by a diagonalization process over all $x \in \mathbb{Q}^n$, since $\mathbb{Q}^n$ is countable, shows that we obtain a subsequence which converges for all the metrics d_x. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, we obtain a convergent sequence $(\Lambda_{i_j})_{j\geq 1}$ for D since

$$\sup_{x\in\mathbb{R}^n} d_x = \sup_{x\in\mathbb{Q}^n} d_x.$$

$\square$

Theorem 7.2. *The density function $\Lambda \to \delta(\mathcal{B}(\Lambda)) = \|\chi(\mathcal{B}(\Lambda))\|_1$ is continuous on $(\mathcal{UD}, D)$ and locally constant.*

Proof. Let $\Lambda_0 \in \mathcal{UD}$, $T > 0$ large enough and $0 < \epsilon < 1$. By Lemma 6.2 and the pointwise pairing property Theorem 2.3 iii), any $\Lambda \in \mathcal{UD}$ such that $D(\Lambda, \Lambda_0) < \epsilon$ is such that the number of elements $\#\{\lambda \in \Lambda \mid \lambda \in B(0,T)\}$ of Λ within $B(0,T)$ satisfies the following inequalities:

$$\#\{\lambda \in \Lambda_0 \mid \lambda \in B(0, T-\epsilon/2)\} \leq \#\{\lambda \in \Lambda \mid \lambda \in B(0,T)\}$$

$$\leq \#\{\lambda \in \Lambda_0 \mid \lambda \in B(0, T+\epsilon/2)\}.$$

The density of the system of balls $\mathcal{B}(\Lambda)$ is equal to

$$\delta(\mathcal{B}(\Lambda)) = \limsup_{T\to+\infty} \#\{\lambda \in \Lambda \mid \lambda \in B(0,T)\} \left(\frac{1}{2T}\right)^n.$$

Since the contribution - to the calculation of the density - of the points of Λ_0 which lie in the annulus $B(0, T+\epsilon/2) \setminus B(0, T-\epsilon/2)$ tends to zero when T tends to infinity by Theorem 1.8 in Rogers [R], we deduce that

$$\delta(\mathcal{B}(\Lambda)) = \delta(\mathcal{B}(\Lambda_0)),$$

hence the claim. $\square$

Let us now finish the proof of Theorem 2.4. Since the metric space $(\mathcal{UD}, D)$ is complete by Theorem 2.3, the subsequence $(\Lambda_{i_j})_{j\geq 1}$ given by Lemma 7.1 is such that there exists a limit point set

$$\Lambda = \lim_{j\to+\infty} \Lambda_{i_j} \in \mathcal{UD}$$

which satisfies, by Theorem 7.2,

$$\delta_n = \lim_{j\to+\infty} \delta(\mathcal{B}(\Lambda_{i_j})) = \delta(\mathcal{B}(\Lambda)).$$

Contradiction.

Let us remark that, in this proof, we did not need assume that the elements Λ_{i_j} are saturated (the same Remark holds for m-saturation).

Let us prove Theorem 2.5. From Theorem 2.4 there exists at least one element of $\mathcal{UD}$, say Λ, of density the packing constant δ_n. Let us assume that there is no completely saturated packing of equal balls of density δ_n and let us show the contradiction. In particular we assume that Λ is not completetly saturated.

Then there would exist an application $i \to m_i$ from $\mathbb{N} \setminus \{0\}$ to $\mathbb{N} \setminus \{0\}$ and a non-stationary sequence $(\Lambda_i)_{i\geq 1}$ such that

(i) $\Lambda_i \in \mathcal{UD}$ with $\Lambda_1 = \Lambda$,

(ii) Λ_{i+1} is obtained from Λ_i by removing m_i balls and placing $m_i + 1$ balls in the holes formed by this removal process,

(iii) $\delta(\mathcal{B}(\Lambda_i)) = \delta_n$ for all $i \geq 1$.

This corresponds to a constant adding of new balls by (ii), but since the density of $\mathcal{B}(\Lambda_1)$ is already maximal, equal to δ_n, this process ocurs at constant density (iii).

As in the proof of Theorem 2.4, we can extract from the sequence $(\Lambda_i)_{i\geq 1}$ a subsequence $(\Lambda_{i_j})_{j\geq 1}$ which is a Cauchy sequence for D. Since $(\mathcal{UD}, D)$ is complete, there exists $\Lambda \in \mathcal{UD}$ such that

$$\Lambda = \lim_{j\to+\infty} \Lambda_{i_j}.$$

The contradiction comes from the pointwise pairing property (iii) in Theorem 2.3 and the continuity of the density function (Theorem 7.2). Indeed, for all j large enough, $D(\Lambda_{i_j}, \Lambda)$ is sufficiently small so that the pointwise pairing property for D prevents the adding of new balls to Λ_{i_j} whatever their number by the process (ii). Therefore, the subsequence $(\Lambda_{i_j})_{j\geq 1}$ would be stationary, which is excluded by assumption. This gives the claim.

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Gilbert MURAZ
INSTITUT FOURIER
Laboratoire de Mathématiques
UMR5582 (UJF-CNRS)
BP 74
38402 St MARTIN D'HÈRES Cedex (France)
Gilbert.Muraz@ujf-grenoble.fr
Jean-Louis VERGER-GAUGRY
INSTITUT FOURIER
Laboratoire de Mathématiques
UMR5582 (UJF-CNRS)
BP 74
38402 St MARTIN D'HÈRES Cedex (France)
jlverger@ujf-grenoble.fr